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Phase locking and chaotic synchronization in an array of three lasers

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Abstract. A linear array of three lasers that are coupled mutually in space is investigated. It is shown that the phase of the laser fields is locked with intermediate coupling while the laser intensities are totally chaotic and chaotically synchronized. When the intensities of lasers reenter the regime of chaotic synchronization at smaller coupling constant, the laser fields show low degree of phase locking. The phase differences in the fields between three lasers show rich patterns when the coupling is changed.

PACS. 05.45 Xt Synchronization; coupled oscillators – 42.65 Sf Dynamics of nonlinear optical systems; optical instabilities, optical chaos and complexity, and optical spatio-temporal dynamics

1 Introduction

Due to its role in understanding complex dynamics and in view of practical applications, theoretical and experimental investigations of chaotic synchronization in coupled nonlinear systems have attracted much attention in recent years [1–16]. The synchronization of chaos was studied in the context of electronic circuits [1–4], Josephson junctions [5–8], laser dynamics [9–13], and secure communications [14–16]. Meanwhile, the generalized synchronization [17–21] and phase synchronization [22–28] in coupled nonlinear systems were also studied.

In most of the previous studies concerning chaotic synchronization in coupled laser systems, only the chaotic intensities are considered [9–16]. However, the phase relations between lasers in the systems also need to be investigated when the lasers are employed in practical applications.

In this paper, the phase locking and chaotic synchronization of intensities in an array of three spatially coupled, solid state lasers with pump modulation is investigated theoretically. In Section 2, theoretical models of three lasers with nearest neighbor coupling are presented. In Section 3, the rich patterns between phase differences of pairs of lasers are investigated together with the intensity relations between three lasers. The periodic synchronization of intensities between three lasers with very strong and extremely weak coupling is studied in Section 4. A discussion of these phenomena concludes the paper.

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2 Theoretical models

The equations describing the time evolution of the complex, slowly varying electric field E_j and gain G_j of laser j of three spatially coupled, pump modulated solid state lasers are as follows [9–12]

$$\begin{aligned}\frac{\mathrm{d}E_j}{\mathrm{d}t} &= \frac{1}{\tau_\mathrm{c}}\left[(G_j-\alpha_j)E_j-\kappa_{j,j+1}E_{j+1}-\kappa_{j,j-1}E_{j-1}\right]+\mathrm{i}\omega_j E_j\\ \frac{\mathrm{d}G_j}{\mathrm{d}t} &= \frac{1}{\tau_\mathrm{f}}\left\{p_j[1+M\sin(\Omega t)]-G_j(1+|E_j|^2)\right\}\cdot \end{aligned}\qquad (1)$$

In these equations, τ_c is the cavity round trip time, τ_f is the fluorescence time, p_j, α_j and ω_j are the pump parameter, loss, and detuning from common cavity mode respectively, M is the modulation depth of the pump beams, and $\Omega = 2\pi f_0$ is the angular frequency of modulation.

The lasers on a straight line are spatially coupled to nearest neighbors with strength κ_{jm} that is assumed to be small. For a laser beam of Gaussian intensity profile with a beam waist ω_0 at $I/I_0 = 1/\mathrm{e}^2$, the coupling strength is defined as

$$\kappa_{jm} = \exp\left[-\frac{(\Delta d_{jm})^2}{2\omega_0^2}\right] \qquad (2)$$

where $\Delta d_{jm} = d_j - d_m$ is the distance between the nearest lasers. This means that small Δd_{jm} corresponds to strong coupling while large Δd_{jm} corresponds to weak coupling. It is obvious that the outer most lasers are not coupled, *i.e.*, $\kappa_{j,j-1} = 0$ for $j = 1$ and $\kappa_{j,j+1} = 0$ for $j = 3$ in a linear array of three lasers.

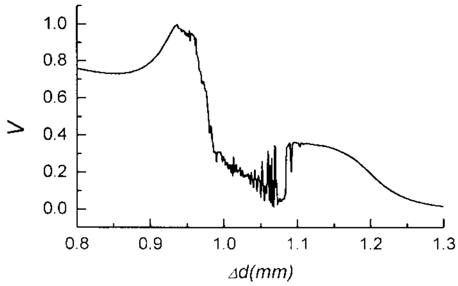


Fig. 1. The average visibility V as a function of separation Δd of the laser system.

The phase and intensity of the laser system can be written as

$$\theta_j = \arctan\left(\frac{E_{yj}}{E_{xj}}\right) \qquad (j = 1, 2, 3)$$
$$I_j = E_{xj}^2 + E_{yj}^2. \tag{3}$$

To characterize the phase dynamics, the average visibility, *i.e.*, the first order coherence V in coupled three lasers is introduced

$$V = \sum_{j,m=1,j\neq m}^{N} \frac{E_j E_m^*}{\sum_{j=1}^{N} I_j} \tag{4}$$

with $N = 3$. In laser systems, $V = 1.0$ corresponds to perfect coherence while $V = 0.0$ corresponds to incoherent light. For $0.0 < V < 1.0$, the light in the array is partially coherent. The light field can be said to be incoherent if $V < 0.05$.

The numerical simulations are based on integration of equations (1) for identical laser system. In numerical calculations, $p_1 = p_2 = p_3$, $\alpha_1 = \alpha_2 = \alpha_3$, τ_c, τ_f, M, f_0 and ω_0 are fixed constants. The distances between lasers are the same, *i.e.*, $\Delta d = \Delta d_{jm}$ and $\kappa = \kappa_{jm}$. The detuning between each laser is given by $\Delta\omega = \omega_{j+1} - \omega_j = 5.0 \times 10^5$ (1/s)$(j = 1, 2)$ with $\omega_1 = 0.0$. Other parameters are varied according to the requirement of chaotic synchronization.

The average visibility V of the laser system is plotted in Figure 1. It is seen that there is a peak in V that corresponds to perfect coherence with $V = 1.0$ at $\Delta d = 0.937$ mm for strong coupling. When the distance Δd between lasers is reduced for very strong coupling, the value of V is decreased to a minimum value of about 0.730 and then increased smoothly as Δd decreased further for even stronger coupling. When Δd is increased with weak coupling, the value of V is dropped sharply around $\Delta d = 0.98$ mm and oscillates with small amplitudes when Δd is increased from 0.98 mm to 1.09 mm. Then the value of V decreases approximately from 0.362 to 0.0163 as Δd is increased further to 1.30 mm. This means that the light fields in three lasers are changed from partially coherent, perfect coherent, partially coherent again, and to incoherent light as the coupling constant κ_{jm} between lasers is decreased.

3 Phase locking and chaotic synchronization

It is easy to calculate the phase θ_j and intensity I_j of laser j from equation (3) by integration of equation (1). The values of θ_j are limited in the range of $0 \leq \theta_j \leq 2\pi$ since the period of θ_j is 2π. The relations between phase θ_j, intensity I_j are plotted in Figures 2A to 2L for different laser separation Δd together with those of phase differences $\Delta\theta_{jk} = \theta_j - \theta_k$ since the average visibility V is related to the phase differences between lasers.

It is seen that both the relations of phase θ_j and intensity I_j show chaotic motion with two bands structure for medium coupling at $\Delta d = 0.947$ mm with $\kappa = 9.47 \times 10^{-5}$ [Figs. 2A and 2B]. However, the relations of the phase difference $\Delta\theta_{jk}$ between lasers show clear evidence of phase locking [Fig. 2C]. It is seen that the relations between $\Delta\theta_{jk}$ show almost the same patterns of long hexagon. Since slight deviations in the shape and positions appear between hexagons, the average visibility V at $\Delta d = 0.947$ mm is about 0.942 with very good coherence.

For slightly smaller coupling at $\Delta d = 0.96901$ mm with $\kappa = 6.13 \times 10^{-5}$, the phase θ_j shows chaotic motion with two bands structure of wider width [Fig. 2D] while that of laser intensity I_j show chaotic synchronization [Fig. 2E]. The relations between phase difference $\Delta\theta_{jk}$ show regular patterns of three lines [Fig. 2F]. Since the curve in the graph of $\Delta\theta_{23}$ *versus* $\Delta\theta_{12}$ is different in shape and position from others, the average visibility V at $\Delta d = 0.96901$ mm is about 0.765 with relatively high coherence.

For weak coupling at $\Delta d = 1.068$ mm with $\kappa = 7.63 \times 10^{-6}$, the relations of phase θ_j, intensity I_j and phase difference $\Delta\theta_{jk}$ show totally chaotic motion [Figs. 2G–2I]. The average visibility V at $\Delta d = 1.068$ mm is about 0.00929. This means that the laser light is almost incoherent.

For slightly weaker coupling at $\Delta d = 1.069$ mm with $\kappa = 7.46 \times 10^{-6}$, the phase θ_j show totally chaotic motion [Fig. 2J] while that of laser intensity I_j reenter the regime of chaotic synchronization [Fig. 2K]. The relations between phase difference $\Delta\theta_{jk}$ show similar patterns [Fig. 2L] as that in Figure 2F. The average visibility V is about 0.348 with low coherence due to larger differences between $\Delta\theta_{jk}$ but not close to zero.

From the phase relations shown in Figures 2A and 2D with intermediate coupling of high degree of coherence, it is seen that a phase locking between adjacent lasers appears evident. The fact that there is not a straight line in the plots between θ_1 and θ_2 is mainly due to the fact that there is a phase shift in one of the two phases, that determines the two bands structure apparent in the plot. Then combing this structure, with the similar ones between θ_2 and θ_3, θ_3 and θ_1, one can get either "hexagonal-like structure" or "three lines structure" observed in the plots of the phase differences [Figs. 2C and 2F]. Since the two bands structure in the plots of θ_j *vs.* θ_m shown in Figure 2A is cleaner than that in Figure 2D, the visibility V for $\Delta d = 0.947$ mm is higher than that for $\Delta d = 0.96901$ mm. Similar phenomenon is also shown in Figures 2D and 2J. It is seen that the structure shown in the plots of θ_j *vs.* θ_m

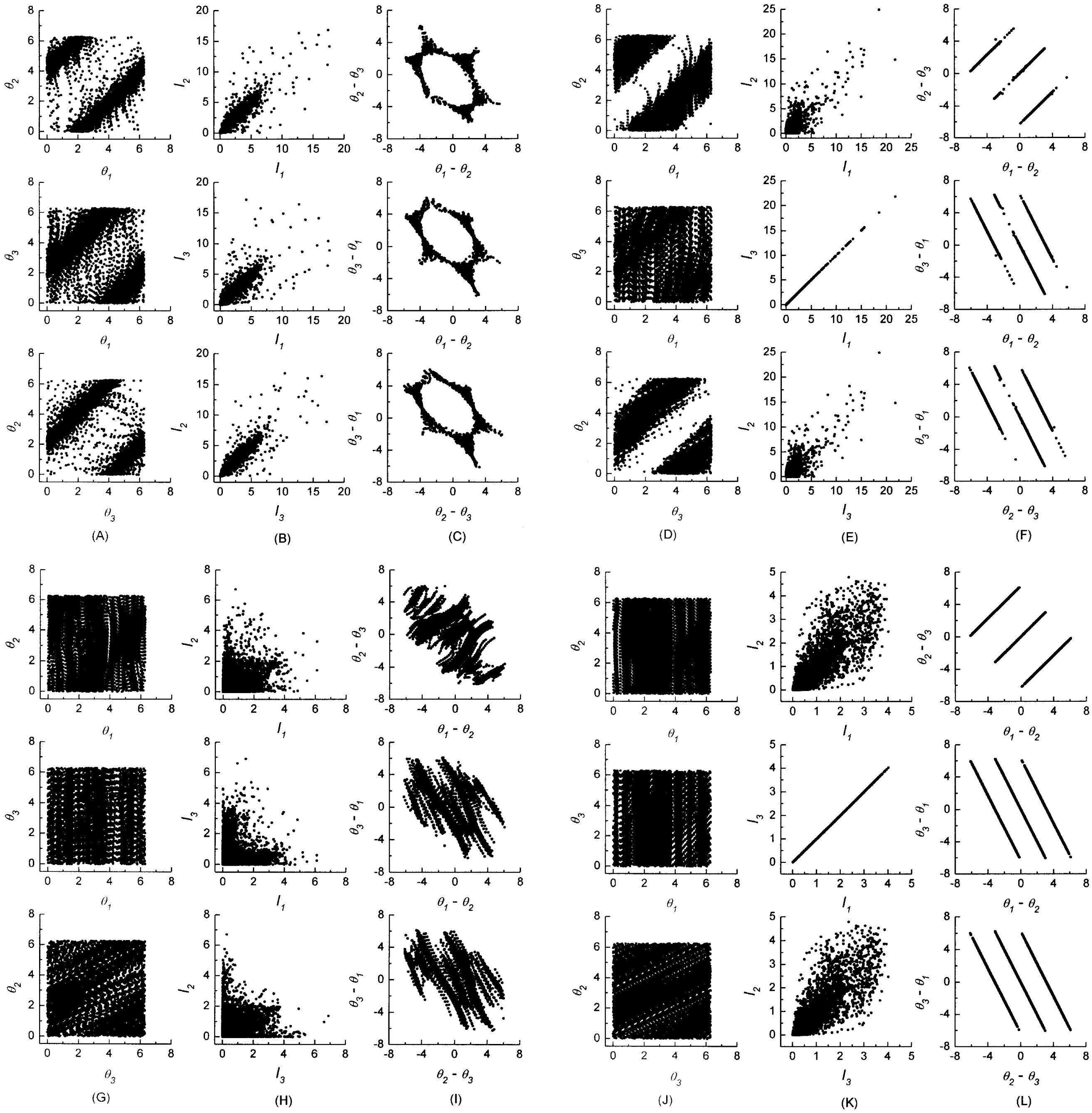


Fig. 2. The relations of phase θ_j, intensity I_j and phase difference $\Delta\theta_{jk}$ between three lasers. (A), (B), (C): $\Delta d = 0.947$ mm; (D), (E), (F): $\Delta d = 0.96901$ mm; (G), (H), (I): $\Delta d = 1.068$ mm; (J), (K), (L): $\Delta d = 1.069$ mm.

can provide some evidence of the degree of phase locking in the laser system.

The normalized power spectrum $P_j(f)$ of the laser field $E_j(t)$ is also calculated by

$$P_j(f) = \frac{|\frac{1}{2\pi}\int E_j(t)\exp(-\mathrm{i}\omega t)\mathrm{d}t|^2}{\int|\frac{1}{2\pi}\int E_j(t)\exp(-\mathrm{i}\omega t)\mathrm{d}t|^2\mathrm{d}\omega} \quad (5)$$

where $\omega = 2\pi f$. If the system is chaotic, the power spectrum is very broad with no particular conspicuous frequencies apparent. Figures 3A to 3D are plots of P_j of the laser fields for $\Delta d = 0.947$ mm, 0.96901 mm, 1.068 mm, and 1.069 mm. It is very clear that $P_j(f)$ is very broad and the lasers are totally chaotic.

4 Two extreme cases

From Figure 1, it is seen that the laser system is highly coherent when the coupling between lasers is very strong with relatively small separation Δd ($\Delta d < 0.98$ mm).

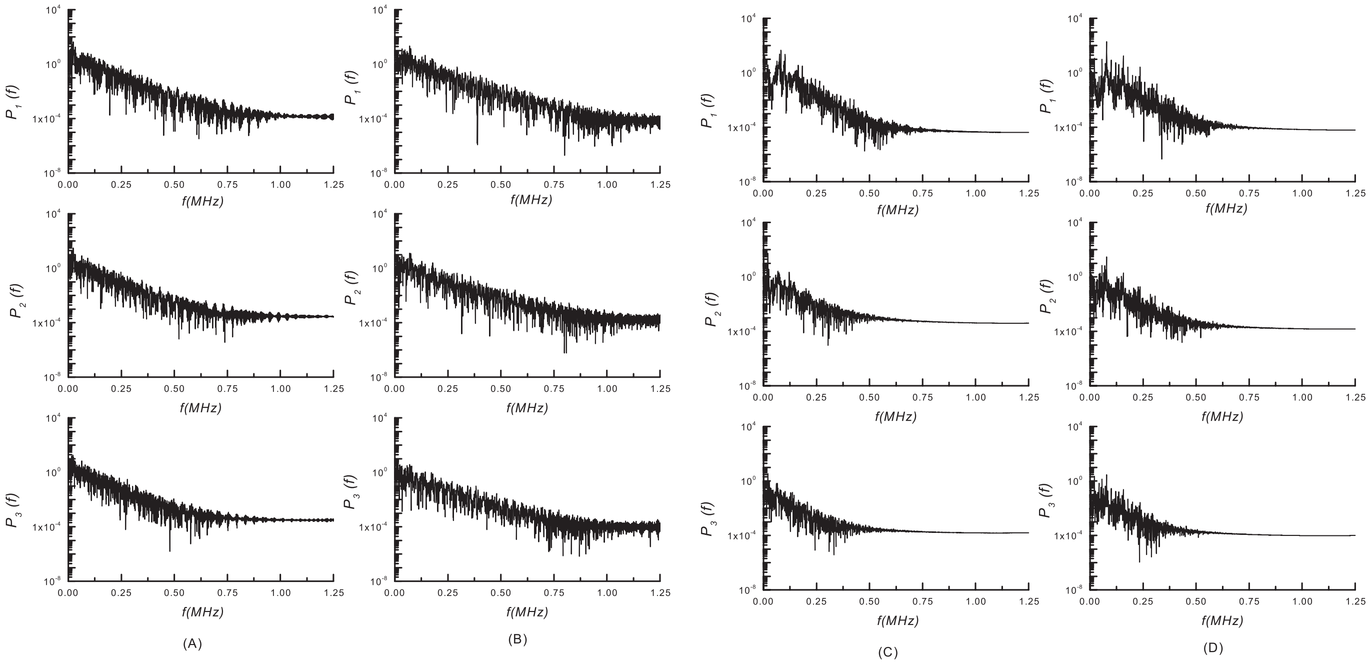


Fig. 3. The normalized power spectrum $P_j(f)$ of the laser system. (A): $\Delta d = 0.947$ mm; (B): $\Delta d = 0.96901$ mm; (C): $\Delta d = 1.068$ mm; (D): $\Delta d = 1.069$ mm.

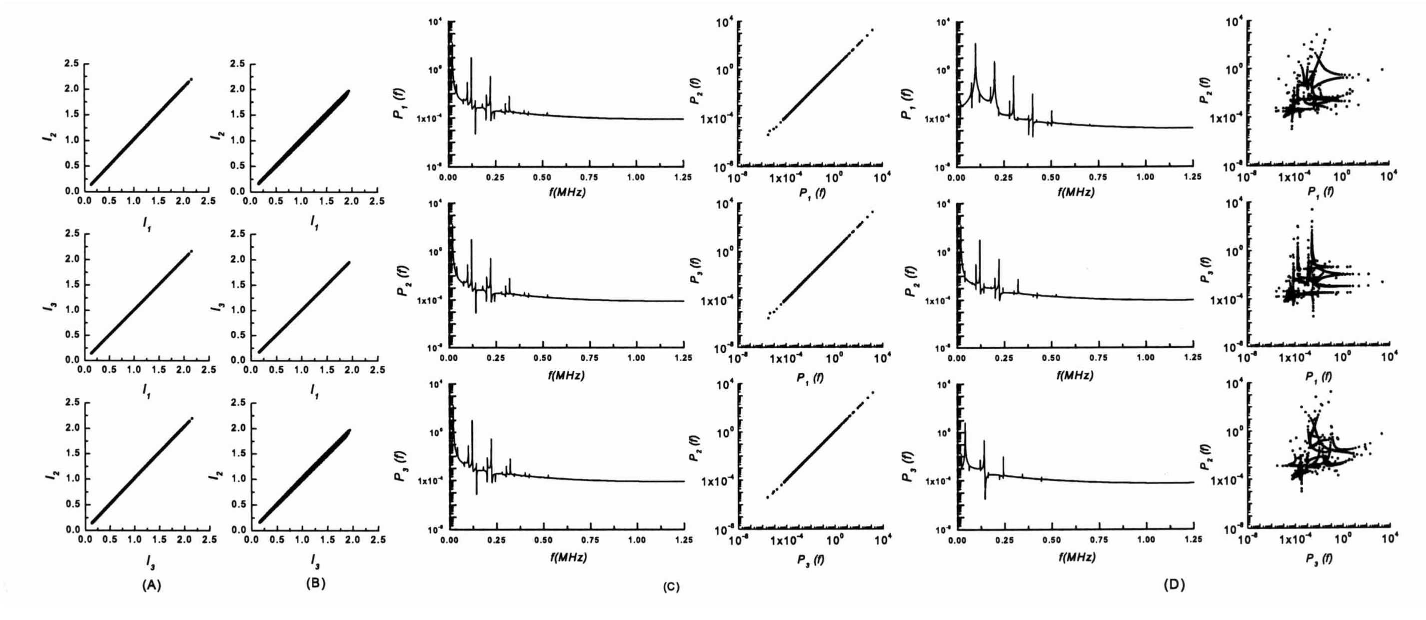
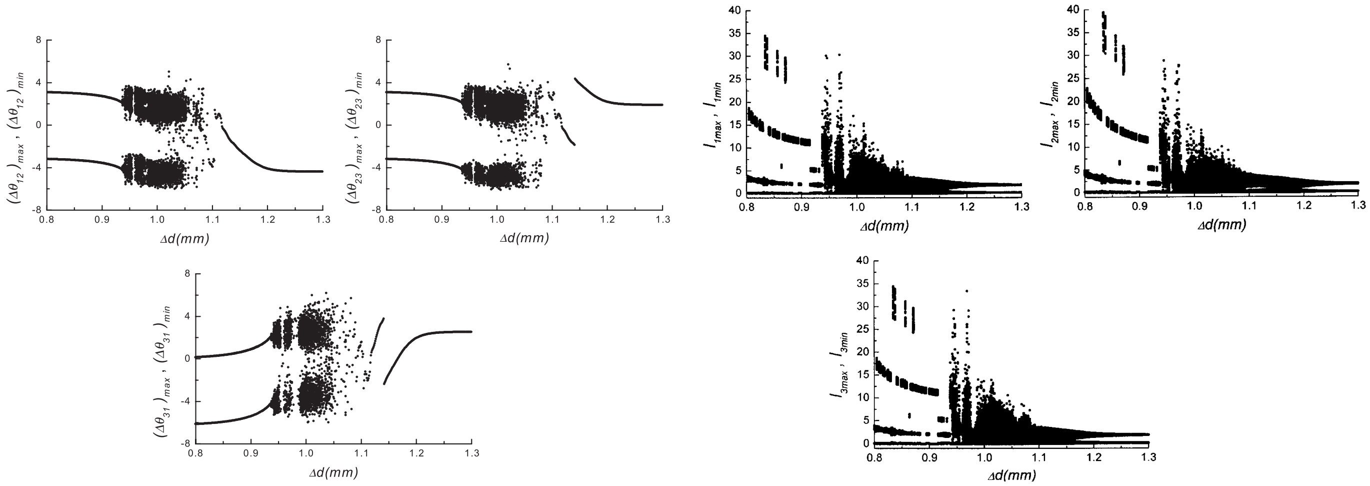


Fig. 4. The intensity $I_j(t)$ and the power spectrum $P_j(f)$ of the laser system. (A), (C): $\Delta d = 0.947$ mm; (B), (D): $\Delta d = 1.30$ mm.

When the coupling between lasers is quite weak with relatively large separation Δd ($\Delta d > 1.20$ mm), the laser system is almost incoherent.

For the two extreme cases of perfect coherent and incoherent light, the separations Δd between lasers are at 0.937 mm and 1.30 mm respectively. The relations between intensity I_j of the three lasers are shown in Figures 4A and 4B for perfect coherence with $V = 1.0$ and for incoherent light with $V = 0.0163$. It is seen that the intensity I_j of the three lasers oscillate periodically for both cases of $\Delta d = 0.937$ mm and 1.30 mm. Though it seems that the intensities are synchronized periodically but not chaotically, there is almost no significant difference between these two extreme cases.

The normalized power spectrums $P_j(f)$ of the laser system for separation between lasers at $\Delta d = 0.937$ mm and 1.30 mm are also plotted in Figures 4C and 4D. It is seen that the three lasers oscillate with the same frequencies and periodic synchronization between any pair of three lasers occurs for strong coupling of $\Delta d = 0.937$ mm with perfect coherence of $V = 1.0$ [Fig. 4C]. While the three lasers oscillate periodically and also independently



(A) (B)

Fig. 5. The bifurcation diagrams of intensity I_j and phase difference $\Delta\theta_{jk}$ of three lasers. (A): $\Delta\theta_{jk}$ *versus* Δd; (B): I_j *versus* Δd.

for very weak coupling of $\Delta d = 1.30$ mm with almost incoherent light of $V = 0.0163$ [Fig. 4D]. The oscillation frequencies are quite different between three lasers and there is no regular relations between any pair of three lasers.

From the intensity I_j in Figures 4A and 4B, it is difficult to distinguish which one is periodically synchronized with perfect coherence of $V = 1.0$. However, from the normalized power spectrum in Figures 4C and 4D, it is clear that the periodic synchronization with perfect coherence of $V = 1.0$ is shown in Figure 4C at laser separation $\Delta d = 0.937$ mm. It seems that the periodic synchronization is mainly due to the strong coupling in the laser system.

5 Discussion

The bifurcation diagrams of phase difference $\Delta\theta_{jk}$ and intensity I_j in an array of three lasers are plotted in Figure 5. It is seen that the phase difference $\Delta\theta_{jk}$ between lasers is locked and the intensity I_j shows periodic motion for strong coupling. While the phase difference is locked, totally chaotic and chaotic synchronization of laser intensities appear for medium coupling in certain parameter regime. Both phase difference and intensities show totally chaotic motion when the coupling is reduced. Then the intensities reenter the regime of chaotic synchronization while the phase difference shows low degree of locking when the coupling decreases further. For extremely small coupling, the lasers in the array oscillate independently while the phases are irrelevant.

When synchronization of chaotic intensity occurs at different distance Δd between lasers, the coherence, *i.e.*, the average visibility V, of the system is quite different. For moderate values of Δd with medium coupling, the coherence of the system is high. While for large values of Δd with weak coupling, the coherence of the system is low but not close to zero.

The phase difference $\Delta\theta_{jk}$ in the fields between three lasers show rich patterns as the values of coupling are changed. It is shown that the chaotic synchronization of intensities can only occur in partially coherent laser light. So the phase dynamics is one of the important conditions for occurrence of chaotic synchronization of intensities in spatially coupled lasers systems [10].

The numerical simulations are also performed for different values of losses with $\alpha_1 < \alpha_2 < \alpha_3$ while other parameters are fixed constants. Similar phenomena, such as chaotic intensity synchronization and phase locking are observed. The disorder introduced in the system can enhance the synchronization in the system. For $\alpha_1 < \alpha_2 < \alpha_3$, slightly mismatch in d_j can still maintain the state of synchronization.

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References

1. L.M. Pecora, T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
2. L.M. Pecora, T.L. Carroll, Phys. Rev. A **44**, 2374 (1991).
3. K.M. Cuomo, A.V. Oppenheim, Phys. Rev. Lett. **71**, 65 (1993).
4. J.F. Heagy, T.L. Carroll, L.M. Pecora, Phys. Rev. A **50**, 1874 (1994).
5. S.H. Strogatz, R.E. Mirollo, Phys. Rev. E **47**, 220 (1993).
6. K. Wisenfeld, P. Colet, S. Strogatz, Phys. Rev. Lett. **76**, 404 (1996).
7. P. Barbara, A.B. Cawthorne, S.V. Shitov, C.J. Lobb, Phys. Rev. Lett. **82**, 1963 (1999).
8. G. Filatrella, N.F. Pedersen, K. Wiesenfeld, Phys. Rev. E **61**, 2513 (2000).

9. H.G. Winful, L. Rahman, Phys. Rev. Lett. **65**, 1575 (1990).
10. R. Roy, K.S. Thornburg Jr, Phys. Rev. Lett. **72**, 2009 (1994).
11. D.Y. Tang, R. Dykstra, N.R. Heckenberg, Phys. Rev. A **54**, 5317 (1996).
12. J.R. Terry, K.S. Thornburg Jr, D.J. DeShazer, G.D. VanWiggeren, S. Zhu, P. Ashwin, R. Roy, Phys. Rev. E **59**, 4036 (1999).
13. K. Otsuka, R. Kawai, S.L. Hwong, J.Y. Ko, J.L. Chern, Phys. Rev. Lett. **84**, 3049 (2000).
14. G.D. VanWiggeren, R. Roy, Science **279**, 1198 (1998).
15. G.D. VanWiggeren, R. Roy, Phys. Rev. Lett. **81**, 3547 (1998).
16. L. Kocarev, U. Parlitz, Phys. Rev. Lett. **74**, 5028 (1995).
17. N.F. Rulkov, M.M. Sushchik, L.S. Tsimiring, H.D.I. Abarbanel, Phys. Rev. E **51**, 980 (1995).
18. H.D.I. Abarbanel, N.F. Rulkov, M.M. Sushchik, Phys. Rev. E **53**, 4528 (1996).
19. L. Kocarev, U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).
20. U. Parlitz, L. Junge, L. Kocarev, Phys. Rev. Lett. **79**, 3158 (1997).
21. R. Brown, Phys. Rev. Lett. **81**, 4835 (1998).
22. M. Rosenblum, A. Pikovsky, J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
23. U. Parlitz, L. Junge, W. Lauterborn, L. Kocarev, Phys. Rev. E **54**, 2115 (1996).
24. M.G. Rosenblum, A.S. Pikovsky, J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).
25. K.J. Lee, Y. Kwak, T.K. Lim, Phys. Rev. Lett. **81**, 321 (1998).
26. Z. Zheng, G. Hu, B. Hu, Phys. Rev. Lett. **81**, 5318 (1998).
27. D.E. Postnov, A.G. Balanov, N.B. Janson, E. Mosekilde, Phys. Rev. Lett. **83**, 1942 (1999).
28. A. Neiman, L. Schimansky-Geier, A. Cornell-Bell, F. Moss, Phys. Rev. Lett. **83**, 4896 (1999).